Stability estimate for hyperbolic inverse problem with time dependent coefficient

Ibtissem Ben Aïcha
Department of Mathematics,
Faculty of Sciences of Bizerte,
7021 Jarzouna Bizerte, Tunisia. *

Abstract

We study the stability in the inverse problem of determining the time dependent zeroth-order coefficient $q(t,x)$ arising in the wave equation, from boundary observations. We derive, in dimension $n \geq 2$, a log-type stability estimate in the determination of q from the Dirichlet-to-Neumann map, in a subset of our domain assuming that it is known outside this subset. Moreover, we prove that we can extend this result to the determination of q in a larger region, and then in the whole domain provided that we have much more data.
Keywords: Inverse problems, Dirichlet-to-Neumann map, Wave equation, Bounded domain, Time dependent potential, X-ray transform, Stability estimate.

1 Introduction

1.1 Statement of the problem

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 2$, with $\mathcal{C}^\infty$ boundary $\Gamma = \partial\Omega$. Given $T > 2\,\mathrm{Diam}(\Omega)$, we introduce the following initial boundary value problem for the wave equation

$$\left\{ \begin{array}{ll} (\partial_t^2 - \Delta + q(t,x))u = 0 & \text{in } Q = [0,T] \times \Omega, \\ u(0,x) = u_0, \ \partial_t u(0,x) = u_1 & \text{in } \Omega, \\ u = f & \text{on } \Sigma = [0,T] \times \Gamma, \end{array} \right. \tag{1.1}$$

where $f \in H^1(\Sigma)$, $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$ and the potential $q \in \mathcal{C}^1(\overline{Q})$ is assumed to be real valued. It is well-known (see [13], [5]) that if the compatibility condition is satisfied, then (1.1) is well-posed. Therefore we can introduce the following operator

$$\begin{array}{rcl} \Lambda_q : H^1(\Sigma) & \longrightarrow & L^2(\Sigma) \\ f & \longmapsto & \partial_\nu u, \end{array}$$

usually called the Dirichlet-to-Neumann map. Here $\nu(x)$ denotes the unit outward normal to Γ at x and $\partial_\nu u$ stands for $\nabla u.\nu$.

*e-mail correspondence: mourad.bellassoued@fsb.rnu.tn

In the present paper, we will first study the inverse problem of recovering the time dependent potential q from the Dirichlet-to-Neumann map Λ_q associated to the problem (1.1) with $(u_0, u_1) = (0, 0)$. This inverse problem is to know whether the knowledge of Λ_q, can uniquely determine the electric time dependent potential q.

Physically, it consists in determining physical properties such as the time evolving density of an inhomogeneous medium by probing it with disturbances generated on the boundary. And the goal is to recover q which describes the property of the medium. We assume that the medium is quiet initially and the Dirichlet data f is a disturbance used to probe it.

The problem of recovering coefficients for hyperbolic equations from boundary measurements was treated by many authors. In [15] Rakesh and Symes proved a uniqueness result in recovering the time independent potential $q(x)$ in the wave equation. In [9] Isakov treated the inverse problem of recovering a zeroth order coefficient and a damping coefficient. These results are concerned in the case where the Dirichlet-to-Neumann map is considered in the whole boundary. A key ingredient in the existing results, is the construction of complex geometric optics solutions concentrating near lines with any direction $\omega \in \mathbb{S}^{n-1}$ and the relationship between the hyperbolic Dirichlet-to-Neumann map and the X-ray transform plays a crucial role. The uniqueness in the determination of time independent potential appearing in the wave equation by a local observations was proved by Eskin [7].

The uniqueness by local measurements is solved well. However, the stability by a local Dirichlet-to-Neumann map is not discussed comprehensively. For it, one can see Bellassoued, Choulі and Yamamoto [3] where a log-type stability estimate was proved in the case where the Neumann data are observed in an arbitrary subdomain of the boundary, Isakov and Sun [11] where a local Dirichlet-to-Neumann map yields an Hölder stability result in determining a coefficient in a subdomain. The case where the Neumann data are observed in the whole boundary, a stability of Hölder type was established in Cipolatti and Lopez [6], Sun [23], and in Riemannian case in M. Bellassoued and D. Dos Santos Ferreira [4], Stefanov and Uhlmann [21].

All the above mentioned results are concerned only with time-independent coefficients. Many authors considered the problem of determining time-dependent coefficients for hyperbolic equations. In [22], Stefanov proved that the time dependent potential $q(t, x)$ arising in the wave equation is uniquely determined from the knowledge of scattering data. In [19], Ramm and Sjöstrand treated the problem of determining the time-dependent potential $q(t, x)$ from Dirichlet-to-Neumann map, on the infinite time-space cylindrical domain $\mathbb{R}_t \times \Omega$, and they proved a uniqueness result under suitable assumptions. In [20], R. Salazar, extended the results in [19] to more general coefficients and proved a result of stability for compactly supported coefficients provided T is sufficiently large.

The inverse problem of determining the time-dependent coefficient $q(t, x)$ from the Dirichlet-to-Neumann map Λ_q, was treated by Ramm and Rakesh [16], they assumed without loss of generality that Ω is a ball and they proved a uniqueness result only in a subset made of lines making $45°$ with the t-axis and meeting the planes $t = 0$ and $t = T$ outside $\overline{Q}$, provided that it's known outside this subset. It's clear that with zero initial data one can not hope to recover $q(t, x)$ over the whole domain Q, even from the knowledge of the full boundary operator Λ_q. This is due to the domain of dependence associated to the hyperbolic problem (1.1) (see [8]). However, in Isakov [10], the ideas from [17]-[18] are used to prove a uniqueness result in determining $q(t, x)$ over the whole domain Q, but he needed much more

information. Indeed his data was the response of the medium for all possible initial data.

In this paper, we will prove a log-type stability estimate which establishes that the time dependent potential $q(t,x)$ depends stably on the Dirichlet-to Neumann map Λ_q in a subset of our domain, provided that it is known outside this subset. After that we prove that we can extend this result to the determination of q in a larger region if we further know the measures $(u(T,.),\partial_t u(T,.))$, where u is the solution of the initial boundary value problem (1.1) with $(u_0,u_1)=(0,0)$. Moreover, we will prove that if our data was the response of the medium for all possible initial data, then we have a log-type stability estimate for this problem over the whole domain Q.

Inspired by the work of M. Bellassoued and D. Dos Santos Ferreira [4], Alden Waters [24] succeeded in proving a type of an Hölder stability estimate for the inverse problem of recovering the X-ray transform of the time-dependent potential q, appearing in the wave equation, from the dynamical Dicrichlet-to Neumann map in Riemannian case. A key ingredient in this result is the construction of Gaussian beam solutions. In the case $n\geq 3$, the inverse problem associated to the system (1.1) with the initial condition $u_0=0$, was treated recently by Y. Kian [12], indeed, inspired by Bellassoued-Jellali-Yamammoto [2]-[1] and using suitable complex geometric optics solutions and Carleman estimate, he proved a log-log type stability estimate in determining the time dependent coefficient $q(t,x)$, from the knowledge of partial Dirichlet-to-Neumann measurement and the measure $u(T,.)$.

Before stating our main results, we recall the following Lemma on the unique existence of a solution to the problem (1.1). The proof is given in [13] (see also [5]).

Lemma 1.1 *Let $T>0$ be given. Suppose that $u_0\in H^1(\Omega)$, $u_1\in L^2(\Omega)$, and $f\in H^1(\Sigma)$. Assume, in addition, that $f(0,.)=u_0|_\Gamma$. Then, there exists a unique solution u of (1.1) satisfying*

$$u\in\mathcal{C}([0,T];\,H^1(\Omega))\cap\mathcal{C}^1([0,T];\,L^2(\Omega)),$$

and there exists $C>0$ such that for any $t\in[0,T]$, we have

$$\|\partial_\nu u\|_{L^2(\Sigma)}+\|u(t,.)\|_{H^1(\Omega)}+\|\partial_t u(t,.)\|_{L^2(\Omega)}\ \leq\ C\left(\|f\|_{H^1(\Sigma)}+\|u_0\|_{H^1(\Omega)}+\|u_1\|_{L^2(\Omega)}\right).$$

From the above Lemma one can see that, if $(u_0,u_1)=(0,0)$, the Dirichlet-to-Neumann map Λ_q is continuous from $H^1(\Sigma)$ to $L^2(\Sigma)$. Therefore we denote by $\|\Lambda_q\|$ its norm in $\mathcal{L}(H^1(\Sigma),\,L^2(\Sigma))$.

1.2 Main results

In order to state our main results we first introduce some notations:

Let $r>0$ such that $T>2r$ and $\overline{\Omega}\subseteq B(0,\frac{r}{2})=\left\{x\in\mathbb{R}^n,\ |x|\leq\dfrac{r}{2}\right\}$. We set $Q_r=[0,T]\times B(0,\frac{r}{2})$. We consider the following sets

$$\mathscr{A}_r=\left\{x\in\mathbb{R}^n,\ \frac{r}{2}<|x|<T-\frac{r}{2}\right\}.$$

$$\mathscr{C}_r^+=\left\{(t,x)\in Q_r,\ |x|<t-\frac{r}{2},\, t>\frac{r}{2}\right\}.$$

$$\mathscr{C}_r^-=\left\{(t,x)\in Q_r,\ |x|<T-\frac{r}{2}-t,\, T-\frac{r}{2}>t\right\}.$$

Note also $Q_r^* = \mathscr{C}_r^+ \cap \mathscr{C}_r^-$. Let denote by $Q_* = Q \cap Q_r^*$. We remark that Q_* is made of lines making $45°$ with the t-axis and meeting the planes $t = 0$ and $t = T$ outside $\overline{Q}_r$. We denote by $Q_\sharp = Q \cap \mathscr{C}_r^+$. We remark that $Q_\sharp$ is made of lines making $45°$ with the t-axis and meeting only the planes $t = 0$ outside $\overline{Q}_r$. Let's note that $Q_* \subset Q_\sharp \subset Q$.

Remark 1 *In the particular case where $\overline{\Omega} = B(0, \frac{r}{2})$, we remark that $Q_* = Q_r^*$ which is the region I in Figure1.2. And $Q_\sharp = \mathscr{C}_r^+$ which is the region $I \cup II \cup III \cup IV$.*

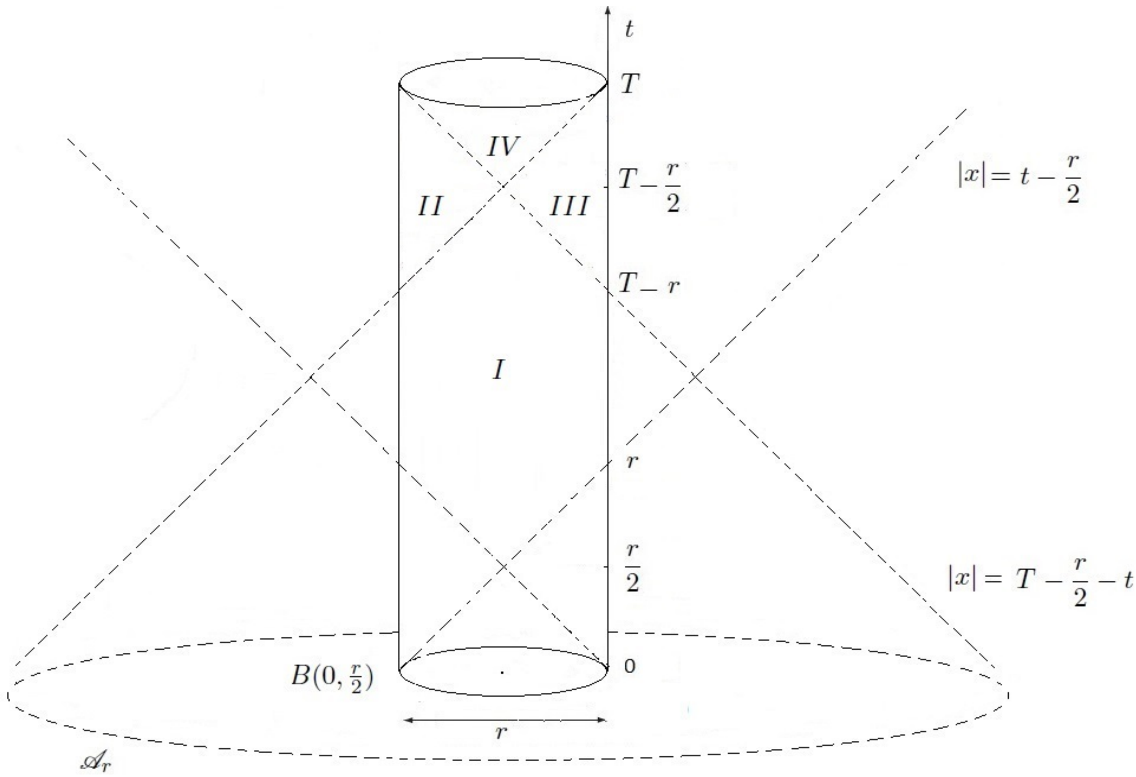

Figure 1.2

Further, given $q_0 \in \mathcal{C}^1(\overline{Q}_r)$ and $M > 0$, we introduce

$$\mathcal{A}^*(q_0, M) = \left\{ q \in \mathcal{C}^1(\overline{Q}_r),\ q = q_0 \text{ in } \overline{Q}_r \setminus Q_*,\ \|q\|_{L^\infty(Q)} \leq M \right\},$$

and

$$\mathcal{A}^\sharp(q_0, M) = \left\{ q \in \mathcal{C}^1(\overline{Q}_r),\ q = q_0 \text{ in } \overline{Q}_r \setminus Q_\sharp,\ \|q\|_{L^\infty(Q)} \leq M \right\}.$$

Then our first main result can be stated as follows:

Theorem 1 *Assume that $T > 2\,\mathrm{Diam}\,(\Omega)$. Then, for every $q_1,\ q_2 \in \mathcal{A}^*(q_0, M)$, there exist two constants $C > 0$ and $\mu_1 \in (0, 1)$, such that we have*

$$\|q_1 - q_2\|_{H^{-1}(Q_*)} \leq C \left(\|\Lambda_{q_1} - \Lambda_{q_2}\|^{\mu_1} + |\log \|\Lambda_{q_1} - \Lambda_{q_2}\||^{-1} \right),$$

where C depends only on $\Omega,\ M,\ T$, and n.

Suppose in addition that $q_1,\, q_2 \in H^{s+1}(Q)$*, for* $s > \dfrac{n}{2}$ *and that* $\|q_i\|_{H^{s+1}(Q)} \le M$*,* $i = 1, 2$*, for some* $M > 0$*, then there exist two constants* $C' > 0$ *and* $\mu_2 \in (0, 1)$ *such that*

$$\|q_1 - q_2\|_{L^\infty(Q_*)} \le C' \left(\|\Lambda_{q_1} - \Lambda_{q_2}\| + |\log \|\Lambda_{q_1} - \Lambda_{q_2}\||^{-1}\right)^{\mu_2}. \tag{1.2}$$

As an immediate consequence of Theorem 1, we have the following uniqueness result.

Corollary 1.1 *(Uniqueness) Under the same assumptions, for every* $q_1,\, q_2 \in \mathcal{A}^*(q_0, M)$*, we have the uniqueness*

$$\Lambda_{q_1}(f) = \Lambda_{q_1}(f), \text{ for any } f \in H^1(\Sigma), \text{ imply } \; q_1(t, x) = q_2(t, x),$$

everywhere in Q_**.*

Let us note that in this result we determine the time dependent coefficient q from full boundary measurements Λ_q only in a subset $Q_* \subset Q$, provided that it is known outside of this part.

In order to extend this result to the determination of q in a larger region $Q_\sharp \supset Q_*$ we need more information about the solution u. Namely we need the measures of $(u(T, .), \partial_t u(T, .))$. So, let's introduce the following boundary operator:

$$\begin{array}{rcl} \mathscr{R}_q : H^1(\Sigma) & \longrightarrow & L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega). \\ f & \longmapsto & (\partial_\nu u,\, u(T, .),\, \partial_t u(T, .)) \end{array}$$

From Lemma 1.1, we deduce that, if $(u_0, u_1) = (0, 0)$, the operator $\mathscr{R}_q$ is continuous from $H^1(\Sigma)$ to $L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega)$. We denote by $\|\mathscr{R}_q\|$ its norm in $\mathcal{L}\Big(H^1(\Sigma),\, L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega)\Big)$.

Then, the second result is the following:

Theorem 2 *Assume that* $T > 2\,\mathrm{Diam}\,(\Omega)$*. Then, for every* $q_1,\, q_2 \in \mathcal{A}^\sharp(q_0, M)$*, there exist two constants* $C > 0$ *and* $\mu_1 \in (0, 1)$*, such that we have*

$$\|q_1 - q_2\|_{H^{-1}(Q_\sharp)} \le C \left(\|\mathscr{R}_{q_1} - \mathscr{R}_{q_2}\|^{\mu_1} + |\; \log \|\mathscr{R}_{q_1} - \mathscr{R}_{q_2}\||^{-1}\right),$$

where C *depends only on* Ω*,* M*,* T*, and* n*.*

Suppose in addition that $q_1,\, q_2 \in H^{s+1}(Q)$*, for* $s > \dfrac{n}{2}$ *and that* $\|q_i\|_{H^{s+1}(Q)} \le M$*,* $i = 1, 2$*, for some* $M > 0$*, then there exist two constants* $C' > 0$ *and* $\mu_2 \in (0, 1)$ *such that*

$$\|q_1 - q_2\|_{L^\infty(Q_\sharp)} \le C' \left(\|\mathscr{R}_{q_1} - \mathscr{R}_{q_2}\| + |\; \log \|\mathscr{R}_{q_1} - \mathscr{R}_{q_2}\||^{-1}\right)^{\mu_2}.$$

where C' *depends on* Ω*,* M*,* T*, and* n*.*

As an immediate consequence of Theorem 2, we have the following uniqueness result.

Corollary 1.2 *(Uniqueness) Under the same assumptions, for every* $q_1,\, q_2 \in \mathcal{A}^\sharp(q_0, M)$*, we have the uniqueness*

$$\mathscr{R}_{q_1}(f) = \mathscr{R}_{q_2}(f), \text{ for any } f \in H^1(\Sigma), \text{ imply } \; q_1(t, x) = q_2(t, x),$$

everywhere in $Q_\sharp$*.*

With zero initial data there is no hope to recover $q(t,x)$ over the whole domain Q, even from the knowledge of the boundary operator $\mathscr{R}_q$. However, from measurements made for all possible initial data, we can extend the results in Theorem 1 and Theorem 2 to the determination of q over the whole domain. We define the boundary operator

$$\begin{array}{rcl}\mathcal{I}_q : H^1(\Sigma)\times H^1(\Omega)\times L^2(\Omega) & \longrightarrow & L^2(\Sigma)\times H^1(\Omega)\times L^2(\Omega). \\ (f,u_0,u_1) & \longmapsto & (\partial_\nu u,\ u(T,.),\ \partial_t u(T,.))\end{array}$$

From Lemme 1.1, we deduce that the linear operator $\mathcal{I}_q$ is continuous from $H^1(\Sigma)\times H^1(\Omega)\times L^2(\Omega)$ to $L^2(\Sigma)\times H^1(\Omega)\times L^2(\Omega)$. We denote by $\|\mathcal{I}_q\|$ its norm.

Then, our last result can be stated as follows:

Theorem 3 *Assume that $T > 2\,\mathrm{Diam}\,(\Omega)$. Then, for every $q_1,\ q_2 \in \mathcal{C}^1(\overline{Q})$, such that $\|q_i\|_{L^\infty(Q)} \leq M$, for $i = 1,\ 2$. There exist two constants $C > 0$ and $\mu_1 \in (0,1)$, such that we have*

$$\|q_1 - q_2\|_{H^{-1}(Q)} \leq C\left(\|\mathcal{I}_{q_1} - \mathcal{I}_{q_2}\|^{\mu_1} + |\log\|\mathcal{I}_{q_1} - \mathcal{I}_{q_2}\||^{-1}\right),$$

where C depends only on $\Omega,\ M,\ T,$ and n.

Suppose in addition that $q_1,\ q_2 \in H^{s+1}(Q)$, for $s > \frac{n}{2}$ and $\|q_i\|_{H^{s+1}(Q)} \leq M$, $i = 1,\ 2$, for some $M > 0$, then there exist two constants $C' > 0$ and $\mu_2 \in (0,1)$ such that

$$\|q_1 - q_2\|_{L^\infty(Q)} \leq C'\left(\|\mathcal{I}_{q_1} - \mathcal{I}_{q_2}\| + |\log\|\mathcal{I}_{q_1} - \mathcal{I}_{q_2}\||^{-1}\right)^{\mu_2}.$$

As an immediate consequence of Theorem 3, we have:

Corollary 1.3 *Under the same assumptions as in Theorem 3, we have the uniqueness*

$$\mathcal{I}_{q_1} = \mathcal{I}_{q_2},\ \textit{imply}\ q_1(t,x) = q_2(t,x),\ \textit{in}\ Q.$$

This paper is organized as follows. In section 2 we construct special optics geometrical solutions to the wave equation (1.1). Using these geometric optics solutions, in section 3 we prove Theorem 1, in section 4 we prove Theorem 2 and in section 5 we prove Theorem 3.

2 Geometric optics solutions

In the present section, we collect some results which are needed in the proof of our main results. We start by the following Lemma (see [13], [5]):

Lemma 2.1 *Let $T > 0$ and $q \in L^\infty(Q)$, suppose that $F \in L^1(0,T;L^2(\Omega))$. The unique solution u of the system*

$$\left\{\begin{array}{ll}\left(\partial_t^2 - \Delta + q(t,x)\right)u(t,x) = F(t,x) & \textit{in } Q,\\ u(0,x) = \partial_t u(0,x) = 0 & \textit{in } \Omega,\\ u(t,x) = 0 & \textit{on } \Sigma,\end{array}\right.$$

satisfies

$$u \in \mathcal{C}([0,T];H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega)).$$

Moreover, there exists a constant $C > 0$ such that

$$\|\partial_t u(t,.)\|_{L^2(\Omega)} + \|\nabla u(t,.)\|_{L^2(\Omega)} \leq C\|F\|_{L^1(0,T;L^2(\Omega))}. \tag{2.3}$$

Using Lemma 2.1 we are able to construct suitable geometrical optics solutions for our inverse problem, which are key ingredients to the proof of our main results.

Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Notice that for all $\omega \in \mathbb{S}^{n-1} = \{\omega \in \mathbb{R}^n,\ |\omega| = 1\}$, the function

$$a(t,x) = \varphi(x + t\omega) \tag{2.4}$$

solves the transport equation

$$(\partial_t - \omega.\nabla)a(t,x) = 0. \tag{2.5}$$

Let's now prove the following Lemma:

Lemma 2.2 *Let $q \in \mathcal{C}^1(\overline{Q})$ such that $\|q\|_{L^\infty(Q)} \leq M$. For $\omega \in \mathbb{S}^{n-1}$, and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we consider the function a defined by (2.4). Then, for $\lambda > 0$, the equation*

$$\left(\partial_t^2 - \Delta + q(t,x)\right) u(t,x) = 0 \textit{ in } Q, \tag{2.6}$$

admits a solution

$$u^\pm \in \mathcal{C}([0,T]; H^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega)),$$

of the following form

$$u^\pm(t,x) = a(t,x)e^{\pm\, i\,\lambda(x.\omega+t)} + R^\pm(t,x), \tag{2.7}$$

where $R^\pm(t,x)$ satisfies

$$R^\pm(t,x) = 0, \ \textit{for all } (t,x) \in \Sigma$$

and

$$\partial_t R^+(0,x) = R^+(0,x) = 0, \ \ x \in \Omega,$$
$$\partial_t R^-(T,x) = R^-(T,x) = 0, \ \ x \in \Omega.$$

Moreover,

$$\lambda\, \|R^\pm\|_{L^2(Q)} + \|\nabla R^\pm\|_{L^2(Q)} \leq C\, \|\varphi\|_{H^3(\mathbb{R}^n)}, \tag{2.8}$$

where C depends only on Ω, T and M.

Proof . We adapt the strategy developed in the proof of a similar result in [15] , where a time independent potential q was considered. In light of (2.6) and (2.7) it is enough to prove the existence of $R^\pm$ satisfying

$$\left\{\begin{array}{ll} \left(\partial_t^2 - \Delta + q(t,x)\right) R^\pm(t,x) = -\left(\partial_t^2 - \Delta + q(t,x)\right)\Big(a(t,x)e^{\pm i\,\lambda\,(x.\omega+t)}\Big) & \text{in } Q,\\ R^\pm(\theta,x) = 0,\ \partial_t R^\pm(\theta,x) = 0, \qquad \theta = 0, \text{ or } T & \text{in } \Omega,\\ R^\pm(t,x) = 0 & \text{on } \Sigma, \end{array}\right. \tag{2.9}$$

and obeying (2.8). We prove the result for u^+. The existence of u^-, being handled in a similar way. To do that note

$$g(t,x) = -\left(\partial_t^2 - \Delta + q(t,x)\right)\Big(a(t,x)e^{i\,\lambda\,(x.\omega+t)}\Big)$$

and use (2.5), getting

$$g(t,x) = -e^{i\,\lambda\,(x.\omega+t)}\left(\partial_t^2 - \Delta + q(t,x)\right) a(t,x) = -e^{i\,\lambda\,(x.\omega+t)} g_0(t,x), \tag{2.10}$$

where $g_0 \in L^1(0,T; L^2(\Omega))$. Thus, R is a suitable solution to the system (2.9) satisfying

$$R \in \mathcal{C}([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$$

and the function

$$w(t,x)=\int_0^t R(s,x)\,ds \tag{2.11}$$

solves the following equation

$$\begin{cases} \left(\partial_t^2-\Delta+q(t,x)\right)w(t,x)=F_1(t,x)+F_2(t,x) & \text{in } Q,\\ w(0,x)=0,\ \partial_t w(0,x)=0 & \text{in } \Omega,\\ w(t,x)=0 & \text{on } \Sigma. \end{cases}$$

Where

$$F_1(t,x)=\int_0^t g(s,x)\,ds,\ \text{ and }\ F_2(t,x)=\int_0^t [q(t,x)-q(s,x)]R(s,x)\,ds. \tag{2.12}$$

Let $\tau\in[0,T]$. In use of Lemma 2.1 on the interval $[0,\tau]$, there exists a constant $C>0$ such that

$$\begin{aligned} \|\partial_t w(\tau,.)\|^2_{L^2(\Omega)} &\leq C\|F_1+F_2\|^2_{L^1(0,\tau;L^2(\Omega))}\\ &\leq C\left(\|F_1\|^2_{L^2(Q)}+\|F_2\|^2_{L^2(0,\tau;L^2(\Omega))}\right). \end{aligned} \tag{2.13}$$

Using (2.11), we have

$$\|F_2\|^2_{L^2(0,\tau,L^2(\Omega))} \leq C_T\|q\|^2_{L^\infty(Q)}\int_0^\tau \|\partial_t w(s,.)\|^2_{L^2(\Omega)}ds.$$

Then, it follows from (2.13) that

$$\|\partial_t w(\tau,.)\|^2_{L^2(\Omega)}\leq C\left(\|F_1\|^2_{L^2(Q)}+\|q\|^2_{L^\infty(Q)}\int_0^\tau \|\partial_t w(s,.)\|^2_{L^2(\Omega)}\,ds\right).$$

Then, from Gronwall's inequality, one gets

$$\|\partial_t w(\tau,.)\|^2_{L^2(\Omega)}\leq C_T\,\|F_1\|^2_{L^2(Q)},$$

where the constant $C_T>0$ depends on T and $\|q\|_{L^\infty}$. From where we get

$$\|R\|^2_{L^2(Q)}\leq C_T\,\|F_1\|^2_{L^2(Q)}, \tag{2.14}$$

according to (2.11). Further, as

$$\|F_1\|^2_{L^2(Q)}=\frac{1}{\lambda^2}\int_Q |\int_0^t g_0(s,x)\,\partial_s(e^{i\lambda(x.\omega+s)})\,ds|^2\,dx\,dt,$$

by (2.10) and (2.12). Then, integrating by parts with respect to s, we deduce from (2.14) that there exists a constant $C>0$ such that

$$\|R\|_{L^2(Q)}\leq \frac{C}{\lambda}\,\|\varphi\|_{H^3(\mathbb{R}^n)}.$$

Finally, Since $\|g\|_{L^2(Q)}\leq C\,\|\varphi\|_{H^3(\mathbb{R}^n)}$, using the energy estimate (2.3) for the problem (2.9) we obtain

$$\|\nabla R\|_{L^2(Q)}\leq C_T\,\|\varphi\|_{H^3(\mathbb{R}^n)},$$

This completes the proof. $\square$

3 Proof of Theorem 1

In the present section we will prove a log-type stability estimate in determining q appearing in the initial boundary value problem (1.1) with $(u_0, u_1) = (0, 0)$. The main ingredients of the proof are geometric optics solutions introduced in Section 2 and X-ray transform. We start by considering geometric optics solutions of the form (2.7). We only assume that $\operatorname{supp}\varphi \subset \mathscr{A}_r$, in such a way we have

$$\operatorname{supp}\varphi \cap \Omega = \emptyset, \text{ and } (\operatorname{supp}\varphi \pm T\omega) \cap \Omega = \emptyset, \ \forall\, \omega \in \mathbb{S}^{n-1}.$$

Then we have the following preliminary estimate which relates the differential of two potentials to the Dirichlet-to-Neumann map.

Lemma 3.1 *Let $q_1, q_2 \in \mathcal{A}^*(q_0, M)$, and put $q = (q_2 - q_1)$. There exists $C > 0$, such that for any $\omega \in \mathbb{S}^{n-1}$ and $\varphi \in \mathcal{C}_0^\infty(\mathscr{A}_r)$, the following estimate*

$$|\int_0^T \int_{\mathbb{R}^n} q(t, x - t\omega)\, \varphi^2(x)\, dx\, dt| \leq C\left(\lambda^3 \|\Lambda_{q_2} - \Lambda_{q_1}\| + \frac{1}{\lambda}\right) \|\varphi\|^2_{H^3(\mathbb{R}^n)}. \tag{3.15}$$

holds true for any sufficiently large $\lambda > 0$.

Proof . In view of Lemma 2.2 and using the fact that supp $\varphi \cap \Omega = \emptyset$, there exists a geometrical optics solutions $u_{2,\lambda}$ to the equation

$$\left(\partial_t^2 - \Delta + q_2(t,x)\right) u_{2,\lambda}(t,x) = 0 \text{ in } Q, \ \ u_{2,\lambda|t=0} = \partial_t u_{2,\lambda|t=0} = 0 \text{ in } \Omega,$$

of the form

$$u_{2,\lambda}(t,x) = a(t,x)e^{i\lambda(x.\omega+t)} + R_{2,\lambda}(t,x), \tag{3.16}$$

where $R_{2,\lambda}$ satisfies

$$\partial_t R_{2,\lambda|t=0} = R_{2,\lambda|t=0} = 0, \quad R_{2,\lambda|\Sigma} = 0.$$

and

$$\|R_{2,\lambda}\|_{L^2(Q)} \leq \frac{C}{\lambda} \|\varphi\|_{H^3(\mathbb{R}^n)}. \tag{3.17}$$

We denote by u_1, the solution of

$$\begin{cases} \left(\partial_t^2 - \Delta + q_1(t,x)\right) u_1(t,x) = 0 & \text{in } Q, \\ u_1(0,x) = \partial_t u_1(0,x) = 0 & \text{in } \Omega, \\ u_1(t,x) = u_{2,\lambda}(t,x) := f_\lambda(t,x), & \text{on } \Sigma. \end{cases}$$

Putting $u(t,x) = u_1(t,x) - u_{2,\lambda}(t,x)$, we get that

$$\begin{cases} \left(\partial_t^2 - \Delta + q_1(t,x)\right) u(t,x) = q(t,x)u_{2,\lambda}(t,x) & \text{in } Q, \\ u(0,x) = \partial_t u(0,x) = 0 & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } \Sigma. \end{cases}$$

Applying Lemma 2.2, once more for λ large enough and using the fact that supp $\varphi \pm T\omega \cap \Omega = \emptyset$, we may find a geometrical optic solution v_λ to the backward wave equation

$$\left(\partial_t^2 - \Delta + q_1(t,x)\right) v_\lambda(t,x) = 0, \text{ in } Q, \quad v_{\lambda|t=T} = \partial_t v_{\lambda|t=T} = 0, \text{ in } \Omega,$$

of the form

$$v_\lambda(t,x)=a(t,x)e^{-i\lambda(x.\omega+t)}+R_{1,\lambda}(t,x), \tag{3.18}$$

where $R_{1,\lambda}$ satisfies

$$\partial_t R_{1,\lambda|t=T}=R_{1,\lambda|t=T}=0,\quad R_{1,\lambda|\Sigma}=0,$$

and

$$\|R_{1,\lambda}\|_{L^2(Q)}\leq \frac{C}{\lambda}\,\|\varphi\|_{H^3(\mathbb{R}^n)}. \tag{3.19}$$

Consequently, by integrating by parts and using the Green's formula, we obtain

$$\begin{aligned}\int_Q q(t,x)u_{2,\lambda}(t,x)v_\lambda(t,x)\,dx\,dt &= \int_Q \left(\partial_t^2-\Delta+q_1(t,x)\right)u(t,x)v_\lambda(t,x)\,dx\,dt\\ &= \int_\Sigma (\Lambda_{q_2}-\Lambda_{q_1})f_\lambda(t,x)v_\lambda(t,x)\,d\sigma\,dt,\end{aligned} \tag{3.20}$$

So, (3.16), (3.18) and (3.20) yield

$$\begin{aligned}\int_Q\ q(t,x)\ a^2(t,x)\,dx\,dt&+\int_Q q(t,x)R_{1,\lambda}(t,x)R_{2,\lambda}(t,x)\,dx\,dt\\ &+\int_Q q(t,x)a(t,x)\left(R_{2,\lambda}(t,x)e^{-i\lambda(x.\omega+t)}+R_{1,\lambda}(t,x)e^{i\lambda(x.\omega+t)}\right)\,dx\,dt\\ =\int_\Sigma\ &(\Lambda_{q_2}-\Lambda_{q_1})f_\lambda(t,x)\ v_\lambda(t,x)\ d\sigma\ dt.\end{aligned} \tag{3.21}$$

From (3.21), (3.17) and (3.19) it follows that

$$|\int_Q q(t,x)\,a^2(t,x)\,dx\,dt|\leq \int_\Sigma |(\Lambda_{q_2}-\Lambda_{q_1})f_\lambda(t,x)\,v_\lambda(t,x)|\,d\sigma\,dt+\frac{C}{\lambda}\,\|\varphi\|^2_{H^3(\mathbb{R}^n)},$$

where the constant $C>0$ does not depend on λ. Hence from the Cauchy-Schwartz inequality and using the fact that $f_\lambda(t,x)=u_{2,\lambda}(t,x)$ on Σ, we obtain

$$|\int_Q q(t,x)a^2(t,x)\,dx\,dt|\leq \|\Lambda_{q_2}-\Lambda_{q_1}\|\,\|u_{2,\lambda}\|_{H^1(\Sigma)}\,\|v_\lambda\|_{L^2(\Sigma)}+\frac{C}{\lambda}\,\|\varphi\|^2_{H^3(\mathbb{R}^n)}, \tag{3.22}$$

Further, as $R_{i,\lambda|\Sigma}=0,$ for $i=1,2$, we deduce from (3.22) that

$$|\int_Q q(t,x)\,a^2(t,x)\,dx\,dt|\leq C\left(\|\Lambda_{q_2}-\Lambda_{q_1}\|\,\|u_{2,\lambda}-R_{2,\lambda}\|_{H^2(Q)}\,\|v_\lambda-R_{1,\lambda}\|_{H^1(Q)}+\frac{1}{\lambda}\,\|\varphi\|^2_{H^3(\mathbb{R}^n)}\right).$$

Bearing in mind that

$$\|v_\lambda-R_{1,\lambda}\|_{H^1(Q)}\leq C\lambda\,\|\varphi\|_{H^3(\mathbb{R}^n)},$$
$$\|u_{2,\lambda}-R_{2,\lambda}\|_{H^2(Q)}\leq C\lambda^2\,\|\varphi\|_{H^3(\mathbb{R}^n)},$$

we end up getting that

$$|\int_Q q(t,x)\,a^2(t,x)\,dx\,dt|\leq C\left(\lambda^3\|\Lambda_{q_2}-\Lambda_{q_1}\|+\frac{1}{\lambda}\right)\,\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

Therefore by extending $q(x,t)$ by zero outside Q_r and recalling (2.4), we find out that

$$|\int_0^T\int_{\mathbb{R}^n} q(t,x-t\omega)\,\varphi^2(x)\,dx\,dt|\leq C\left(\lambda^3\|\Lambda_{q_2}-\Lambda_{q_1}\|+\frac{1}{\lambda}\right)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

This completes the proof of the Lemma. □

3.1 X-ray transform

The X-ray transform R maps a function in $\mathbb{R}^{n+1}$ into the set of its line integrals. More precisely, if $\omega \in \mathbb{S}^{n-1}$ and $(t,x) \in \mathbb{R}^{n+1}$,

$$R(f)(\omega, x) := \int_{\mathbb{R}} f(t, x - t\omega)\, dt,$$

is the integral of f over the lines $\{(t, x - t\omega),\ \ t \in \mathbb{R}\}$.

Using the above Lemma, we can estimate the X-ray transform of the differential of potentials as follows:

Lemma 3.2 *There exists a constant $C > 0$, $\beta > 0$, $\delta > 0$, and $\lambda_0 > 0$ such that for all $\omega \in \mathbb{S}^{n-1}$, we have*

$$|R(q)(\omega, y)| \leq C \left(\lambda^{\beta} \|\Lambda_{q_2} - \Lambda_{q_1}\| + \frac{1}{\lambda^{\delta}} \right), \quad \textit{a.e. } y \in \mathbb{R}^n.$$

for any $\lambda \geq \lambda_0$.

Proof . Let $\phi \in \mathcal{C}_0^{\infty}(\mathbb{R}^n)$ be a positive function which is supported in the unit ball $B(0,1)$ such that $\|\phi\|_{L^2(\mathbb{R}^n)} = 1$. Define

$$\varphi_\varepsilon(x) = \varepsilon^{-n/2} \phi \left(\frac{x-y}{\varepsilon} \right)$$

where $y \in \mathscr{A}_r$. Then for sufficiently small $\varepsilon > 0$ we can verify that

$$\operatorname{supp} \varphi_\varepsilon \cap \Omega = \emptyset, \quad \text{and} \quad \operatorname{supp} \varphi_\varepsilon \pm T\omega \cap \Omega = \emptyset.$$

And we have

$$\begin{aligned} |\int_0^T q\,(t, y - t\omega)\, dt| &= |\int_0^T \int_{\mathbb{R}^n} q(t, y - t\omega)\, \varphi_\varepsilon^2(x)\, dx\, dt\,| \\ &\leq \ \ |\int_0^T \int_{\mathbb{R}^n} q(t, x - t\omega)\, \varphi_\varepsilon^2(x)\, dx\, dt| + |\int_0^T \int_{\mathbb{R}^n} \left(q(t, y - t\omega) - q(t, x - t\omega) \right) \varphi_\varepsilon^2(x)\, dx\, dt\,|. \end{aligned}$$

Since $\|q\|_{C^1(Q)} \leq M$, we have

$$|q(t, y - t\omega) - q(t, x - t\omega)| \leq C\, |x - y|.$$

Applying Lemma 3.1 with $\varphi = \varphi_\varepsilon$, we obtain

$$|\int_0^T q(t, y - t\omega)\, dt| \leq C \left(\lambda^3 \|\Lambda_{q_2} - \Lambda_{q_1}\| + \frac{1}{\lambda} \right) \|\varphi_\varepsilon\|^2_{H^3(\mathbb{R}^n)} + C \int_{\mathbb{R}^n} |x - y|\, \varphi_\varepsilon^2(x)\, dx. \tag{3.23}$$

On the other hand, we have

$$\|\varphi_\varepsilon\|_{H^3(\mathbb{R}^n)} \leq C\, \varepsilon^{-3}, \quad \int_{\mathbb{R}^n} |x - y|\, \varphi_\varepsilon^2(x)\, dx \leq C\varepsilon.$$

Thus, from (3.23), we obtain

$$|\int_0^T q(t, y - t\omega)\, dt| \leq C \left(\lambda^3 \|\Lambda_{q_2} - \Lambda_{q_1}\| + \frac{1}{\lambda} \right) \varepsilon^{-6} + C\varepsilon.$$

We select ε such that

$$\varepsilon = \frac{\varepsilon^{-6}}{\lambda}.$$

Then there exist constants $\delta > 0$ and $\beta > 0$ such that

$$| \int_0^T q(t, y - t\omega)\, dt | \leq C \left(\lambda^\beta \|\Lambda_{q_2} - \Lambda_{q_1}\| + \frac{1}{\lambda^\delta} \right).$$

Using the fact that $q = 0$, outside Q_r, we get

$$| \int_{\mathbb{R}} q(t, y - t\omega)\, dt | \leq C \left(\lambda^\beta \|\Lambda_{q_2} - \Lambda_{q_1}\| + \frac{1}{\lambda^\delta} \right), \quad \text{a.e.}\, y \in \mathscr{A}_r,\ \omega \in \mathbb{S}^{n-1}. \tag{3.24}$$

On the other hand, if $|y| \leq \dfrac{r}{2}$, then

$$q(t, y - t\omega) = 0 \ \forall\, t \in \mathbb{R}. \tag{3.25}$$

Indeed, we have

$$|y - t\omega| \geq |t| - |y| \geq t - \frac{r}{2}. \tag{3.26}$$

So that, if $t > \dfrac{r}{2}$, from (3.26), we have $(t, y - t\omega) \notin \mathscr{C}_r^+$. And if $t \leq \dfrac{r}{2}$, we have also $(t, y - t\omega) \notin \mathscr{C}_r^+$. Consequently,

$$(t, y - t\omega) \notin \mathscr{C}_r^+ \supset Q_*, \quad \text{for all } t \in \mathbb{R}.$$

Using the fact that $q = q_2 - q_1 = 0$ outside Q_*, we deduce (3.25). Therefore,

$$\int_{\mathbb{R}} q(t, y - t\omega)\, dt = 0, \quad \text{a.e. } y \in B(0, \frac{r}{2}).$$

By a similar way, we prove that in the case where $|y| \geq T - \frac{r}{2}$, we have

$$(t, y - t\omega) \notin \mathscr{C}_r^- \supset Q_*, \quad \text{for all } t \in \mathbb{R}.$$

Then we conclude that

$$\int_{\mathbb{R}} q(t, y - t\omega)\, dt = 0, \quad \text{a.e. } y \notin \mathscr{A}_r,\ \omega \in \mathbb{S}^{n-1}. \tag{3.27}$$

Consequently, by (3.24) and (3.27), one gets

$$|R(q)(\omega, y)| = | \int_{\mathbb{R}} q(t, y - t\omega)\, dt | \leq C \left(\lambda^\beta \|\Lambda_{q_2} - \Lambda_{q_1}\| + \frac{1}{\lambda^\delta} \right), \text{ a.e. } y \in \mathbb{R}^n,\ \omega \in \mathbb{S}^{n-1}.$$

This completes the proof of the Lemma. $\square$

Let now

$$E = \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n, \ |\tau| \leq |\xi|\},$$

and let the Fourier transform of $q \in L^1(\mathbb{R}^{n+1})$

$$\widehat{q}(\tau, \xi) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} q(x, t) e^{-ix.\xi} e^{-it\tau}\, dx\, dt.$$

Our goal now is to prove the following

Lemma 3.3 *There exist constants $C > 0,\ \beta > 0,\ \delta > 0$ and $\lambda_0 > 0$ such that the following estimate holds*

$$|\widehat{q}(\tau,\xi)| \leq C\left(\lambda^{\beta}\|\Lambda_{q_2} - \Lambda_{q_1}\| + \frac{1}{\lambda^{\delta}}\right),$$

for any $(\tau,\xi) \in E$ and $\lambda \geq \lambda_0$.

Proof . Let $(\tau,\xi) \in E$ and $\zeta \in \mathbb{S}^{n-1}$ such that $\xi.\zeta = 0$. By defining

$$\omega = \frac{\tau}{|\xi|^2}.\xi + \sqrt{1 - \frac{\tau^2}{|\xi|^2}}.\zeta,$$

we have $\omega \in \mathbb{S}^{n-1}$ and $\omega.\xi = \tau$.

By the change of variable $x = y - t\omega$ we have for all $\xi \in \mathbb{R}^n,\ \omega \in \mathbb{S}^{n-1}$

$$\begin{aligned}
\int_{\mathbb{R}^n} R(q)(\omega,y)\, e^{-iy.\xi}\, dy &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}} q(t,y-t\omega)\, dt\right) e^{-iy.\xi}\, dy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} q(t,x)\, e^{-ix.\xi} e^{-it(\omega.\xi)}\, dx\, dt \\
&= \widehat{q}(\omega.\xi,\xi) \\
&= \widehat{q}(\tau,\xi).
\end{aligned}$$

Denote $(\tau,\xi) = (\omega.\xi,\xi) \in E$. Since $\operatorname{supp} q(t,.) \subset \Omega \subset B(0,\frac{r}{2})$, then we have

$$\int_{\mathbb{R}^n \cap B(0,\frac{r}{2}+T)} R(q)(\omega,y)\, e^{-iy.\xi}\, dy = \widehat{q}(\tau,\xi).$$

In terms of Lemma 3.2, the proof is completed. □

3.2 Stability estimate

We are now in position to complete the proof of Theorem 1. For $\rho > 0$ and $\gamma \in (\mathbb{N}\cup\{0\})^{n+1}$, we denote

$$|\gamma| = \gamma_1 + ... + \gamma_{n+1}, \qquad B(0,\rho) = \{x \in \mathbb{R}^{n+1},\ |x| < \rho\}.$$

We consider the following Lemma

Lemma 3.4 *(see [25]) Let O be an open set of $B(0,1)$, and F an analytic function in $B(0,2)$, satisfying the following property: there exist constant $M, \eta > 0$ such that*

$$\|\partial^{\gamma} F\|_{L^{\infty}(B(0,2))} \leq \frac{M|\gamma|!}{\eta^{|\gamma|}}, \quad \forall\, \gamma \in (\mathbb{N}\cup\{0\})^{n+1}.$$

Then,

$$\|F\|_{L^{\infty}(B(0,1))} \leq (2M)^{1-\mu}\|F\|^{\mu}_{L^{\infty}(O)}.$$

where $\mu \in (0,1)$ depends on n, η and $|O|$.

The Lemma is conditional stability for the analytic continuation, and see Lavrent'ev, Romanov and Shishat·sKii. [14] for classical results. For fixed $\alpha > 0$, let us set

$$F_\alpha(\tau,\xi) = \widehat{q}(\alpha(\tau,\xi)) \ \text{ for } \ (\tau,\xi) \in \mathbb{R}^{n+1}.$$

It is easily seen that F_α is analytic and we have

$$\begin{aligned} |\partial^\gamma F_\alpha(\tau,\xi)| = |\partial^\gamma \widehat{q}(\alpha(\tau,\xi))| &= |\partial^\gamma \int_{\mathbb{R}^{n+1}} q(t,x)\, e^{-i\alpha(t,x).(\tau,\xi)}\, dx\, dt| \\ &= |\int_{\mathbb{R}^{n+1}} q(t,x)(-i)^{|\gamma|}\alpha^{|\gamma|}(t,x)^\gamma e^{-i\alpha(t,x).(\tau,\xi)}\, dx\, dt|. \quad (3.28) \end{aligned}$$

Therefore, from (3.28) one gets

$$|\partial^\gamma F_\alpha(\tau,\xi)| \le \int_{\mathbb{R}^{n+1}} |q(t,x)|\alpha^{|\gamma|}(|x|^2+t^2)^{\frac{|\gamma|}{2}}\, dx\, dt \le \|q\|_{L^1(Q_*)}\ \alpha^{|\gamma|}\ (2T^2)^{\frac{|\gamma|}{2}} \le C\ \frac{|\gamma|!}{(T^{-1})^{|\gamma|}}\ e^\alpha.$$

Then, applying Lemma 3.4 in the set $O = \mathring{E} \cap B(0,1)$ with $M = Ce^\alpha$ and $\eta = T^{-1}$, we can take a constant $\mu \in (0,1)$ such that

$$|F_\alpha(\tau,\xi)| = |\widehat{q}(\alpha(\tau,\xi))| \le Ce^{\alpha(1-\mu)}\|F_\alpha\|^\mu_{L^\infty(O)}, \qquad (\tau,\xi) \in B(0,1).$$

Hence, by using the fact that $\alpha\,\mathring{E} = \{\alpha(\tau,\xi),\ (\tau,\xi) \in \mathring{E}\} = \mathring{E}$, we get for $(\tau,\xi) \in B(0,\alpha)$

$$\begin{aligned} |\widehat{q}(\tau,\xi)| = |F_\alpha(\alpha^{-1}(\tau,\xi)| &\le Ce^{\alpha(1-\mu)}\, \|F_\alpha\|^\mu_{L^\infty(O)} \\ &\le Ce^{\alpha(1-\mu)}\|\,\widehat{q}\,\|^\mu_{L^\infty(B(0,\alpha)\cap\mathring{E})} \\ &\le Ce^{\alpha(1-\mu)}\|\,\widehat{q}\,\|^\mu_{L^\infty(\mathring{E})}. \quad (3.29) \end{aligned}$$

On the other hand we have

$$\begin{aligned} \|q\|^{2/\mu}_{H^{-1}(\mathbb{R}^{n+1})} &= \left(\int_{|(\tau,\xi)|<\alpha}(1+|(\tau,\xi)|^2)^{-1}|\widehat{q}(\tau,\xi)|^2\, d\tau d\xi + \int_{|(\tau,\xi)|\ge\alpha}(1+|(\tau,\xi)|^2)^{-1}|\widehat{q}(\tau,\xi)|^2\, d\tau d\xi\right)^{1/\mu} \\ &\le C\left(\alpha^{n+1}\ \|\widehat{q}\|^2_{L^\infty(B(0,\alpha))} +\ \alpha^{-2}\ \|q\|^2_{L^2(\mathbb{R}^{n+1})}\right)^{1/\mu}. \end{aligned}$$

From (3.29) and applying Lemma 3.3, we obtain

$$\begin{aligned} \|q\|^{2/\mu}_{H^{-1}(\mathbb{R}^{n+1})} &\le C\left(\alpha^{n+1}\, e^{2\alpha(1-\mu)}\,(\lambda^\beta\|\Lambda_{q_2}-\Lambda_{q_1}\| + \frac{1}{\lambda^\delta})^{2\mu} + \alpha^{-2}\right)^{1/\mu} \\ &\le C\left(\alpha^{\frac{n+1}{\mu}}\, e^{\frac{2\alpha(1-\mu)}{\mu}}\, \lambda^{2\beta}\|\Lambda_{q_2}-\Lambda_{q_1}\|^2 + \alpha^{\frac{n+1}{\mu}}\, e^{\frac{2\alpha(1-\mu)}{\mu}}\, \lambda^{-2\delta} + \alpha^{-2/\mu}\right). \end{aligned}$$

Let $\alpha_0 > 0$ be sufficiently large and $\alpha > \alpha_0$. Set

$$\lambda = \alpha^{\frac{n+3}{2\mu\delta}}\, e^{\frac{\alpha(1-\mu)}{\mu\delta}}.$$

By $\alpha > \alpha_0$, we can assume that $\lambda > \lambda_0$, and we have

$$\alpha^{\frac{n+1}{\mu}}\, e^{\frac{2\alpha(1-\mu)}{\mu}}\, \lambda^{-2\delta} = \alpha^{-2/\mu}.$$

Then

$$\begin{array}{rcl}\|q\|^{2/\mu}_{H^{-1}(\mathbb{R}^{n+1})} & \leq & C\left(\alpha^{\frac{\delta(n+1)+\beta(n+3)}{\delta\mu}}\, e^{\frac{2\alpha(\delta+\beta)(1-\mu)}{\delta\mu}}\|\Lambda_{q_2}-\Lambda_{q_1}\|^2+\alpha^{-2/\mu}\right)\\ & \leq & C\left(e^{N\alpha}\|\Lambda_{q_2}-\Lambda_{q_1}\|^2+\alpha^{-2/\mu}\right),\end{array}$$

where N depends on δ, β, n, and μ. In order to minimize the right hand-side with respect to α, we set

$$\alpha=\frac{1}{N}|\,\log\|\Lambda_{q_2}-\Lambda_{q_1}\|\,|,$$

where we assume that

$$0<\|\Lambda_{q_2}-\Lambda_{q_1}\|<c.$$

It follows that

$$\begin{array}{rcl}\|q\|_{H^{-1}(Q_*)}\leq\|q\|_{H^{-1}(\mathbb{R}^{n+1})} & \leq & C\left(\|\Lambda_{q_2}-\Lambda_{q_1}\|+|\,\log\|\Lambda_{q_2}-\Lambda_{q_1}\|\,|^{-2/\mu}\right)^{\mu/2}\\ & \leq & C\left(\|\Lambda_{q_2}-\Lambda_{q_1}\|^{\mu/2}+|\log\|\Lambda_{q_2}-\Lambda_{q_1}\||^{-1}\right).\end{array}$$

The estimate (1.2), is now an easy consequence of the Sobolev embedding theorem and an interpolation inequality. Let $\delta'>0$ such that $s=n/2+2\delta'$. Then, we have

$$\begin{array}{rcl}\|q\|_{L^\infty(Q_*)} & \leq & C\|q\|_{H^s(Q_*)}\\ & \leq & C\,\|q\|^{1-\beta}_{H^{-1}(Q_*)}\,\|q\|^{\beta}_{H^{s+1}(Q_*)}\\ & \leq & C\,\|q\|^{1-\beta}_{H^{-1}(Q_*)},\end{array}$$

for some $\beta\in(0,1)$. Then the proof of Theorem 1 is completed.

4 Proof of Theorem 2

This section is devoted to the proof of Theorem 2. We will extend the stability estimate (1.2) given in Theorem 1, to an estimate in a larger region $Q_\sharp\supset Q*$. Differently to Theorem 1, here the observations are given by the boundary operator $\mathscr{R}_q$ introduced in Subsection 1.2. We need to consider geometric optics solutions similar to the one used in the previous section, but this time, we will only assume that $\mathrm{supp}\,\varphi\cap\Omega=\emptyset$. (We don't need to assume that $\mathrm{supp}\,\varphi\pm T\omega\cap\Omega=\emptyset$). Let's first recall the definition of the operator $\mathscr{R}_q$:

$$\begin{array}{rcl}\mathscr{R}_q: H^1(\Sigma) & \longrightarrow & L^2(\Sigma)\times H^1(\Omega)\times L^2(\Omega).\\ f & \longmapsto & (\partial_\nu u,\,u(T,.),\,\partial_t u(T,.)).\end{array}$$

We denote by

$$\mathscr{R}^1_{q_j}(f)=\partial_\nu u_j,\quad \mathscr{R}^2_{q_j}(f)=u_{j,}(T,.),\quad \mathscr{R}^3_{q_j}(f)=\partial_t u_j(T,.),\quad \text{for}\,j=1,2.$$

Lemma 4.1 *Let $q_1,q_2\in\mathcal{A}^\sharp(q_0,M)$, $\varphi\in\mathcal{C}^\infty_0(\mathbb{R}^n)$, such that $\mathrm{supp}\,\varphi\cap\Omega=\emptyset$, and put $q=(q_2-q_1)$. Then, there exists $C>0$, such that for any $\omega\in\mathbb{S}^{n-1}$ the following estimate*

$$|\int_0^T\int_{\mathbb{R}^n}q(t,x-t\omega)\,\varphi^2(x)\,dx\,dt|\leq C\,\left(\lambda^3\|\mathscr{R}_{q_2}-\mathscr{R}_{q_1}\|+\frac{1}{\lambda}\right)\,\|\varphi\|^2_{H^3(\mathbb{R}^n)}\qquad(4.30)$$

holds true for any sufficiently large $\lambda>0$.

Proof . In view of Lemma 2.2 and using the fact that supp $\varphi \cap \Omega = \emptyset$, there exists a geometrical optics solutions $u_{2,\lambda}$ to the equation

$$\left(\partial_t^2 - \Delta + q_2(t,x)\right) u_{2,\lambda}(t,x) = 0 \text{ in } Q, \;\; u_{2,\lambda|t=0} = \partial_t u_{2,\lambda|t=0} = 0 \text{ in } \Omega,$$

of the form

$$u_{2,\lambda}(t,x) = a(t,x)e^{i\lambda(x.\omega+t)} + R_{2,\lambda}(t,x), \tag{4.31}$$

where $R_{2,\lambda}$ satisfies

$$\partial_t R_{2,\lambda|t=0} = R_{2,\lambda|t=0} = 0, \quad R_{2,\lambda|\Sigma} = 0,$$

and

$$\|R_{2,\lambda}\|_{L^2(Q)} \leq \frac{C}{\lambda}\,\|\varphi\|_{H^3(\mathbb{R}^n)}. \tag{4.32}$$

We denote by $u_{1,\lambda}$, the solution of

$$\left\{\begin{array}{ll} \left(\partial_t^2 - \Delta + q_1(t,x)\right) u_{1,\lambda}(t,x) = 0 & \text{in } Q, \\ u_{1,\lambda}(0,x) = \partial_t u_{1,\lambda}(0,x) = 0 & \text{in } \Omega, \\ u_{1,\lambda}(t,x) = u_{2,\lambda}(t,x) := f_\lambda(t,x), & \text{on } \Sigma. \end{array}\right.$$

Putting $u_\lambda(t,x) = u_{1,\lambda}(t,x) - u_{2,\lambda}(t,x)$, we get that

$$\left\{\begin{array}{ll} \left(\partial_t^2 - \Delta + q_1(t,x)\right) u_\lambda(t,x) = q(t,x)u_{2,\lambda}(t,x) & \text{in } Q \\ u_\lambda(0,x) = \partial_t u_\lambda(0,x) = 0 & \text{in } \Omega \\ u_\lambda(t,x) = 0 & \text{on } \Sigma. \end{array}\right.$$

Applying Lemma 2.2, once more for λ large enough, we may find a geometrical optic solution v_λ to the backward wave equation

$$\left(\partial_t^2 - \Delta + q_1(t,x)\right) v_\lambda(t,x) = 0, \;\text{ in } Q,$$

of the form

$$v_\lambda(t,x) = a(t,x)e^{-i\lambda(x.\omega+t)} + R_{1,\lambda}(t,x), \tag{4.33}$$

where $R_{1,\lambda}$ satisfies

$$\partial_t R_{1,\lambda|t=T} = R_{1,\lambda|t=T} = 0, \quad R_{1,\lambda|\Sigma} = 0,$$

and

$$\|R_{1,\lambda}\|_{L^2(Q)} \leq \frac{C}{\lambda}\,\|\varphi\|_{H^3(\mathbb{R}^n)}. \tag{4.34}$$

Consequently, by integrating by parts and using the Green's formula we obtain

$$\begin{aligned} \int_Q q(t,x)u_{2,\lambda}(t,x)v_\lambda(t,x)\,dx\,dt \;\;&=\;\; \int_\Sigma (\mathscr{R}^1_{q_2} - \mathscr{R}^1_{q_1})(f_\lambda)v_\lambda(t,x)\,d\sigma\,dt \\ &\quad + \int_\Omega \left(\mathscr{R}^2_{q_2} - \mathscr{R}^2_{q_1}\right)(f_\lambda)\,\partial_t v_\lambda(T,.)\,dx \\ &\quad - \int_\Omega \left(\mathscr{R}^3_{q_2} - \mathscr{R}^3_{q_1}\right)(f_\lambda)\,v_\lambda(T,.)\,dx, \end{aligned} \tag{4.35}$$

Then, by replacing $u_{2,\lambda}$ and v_λ by their expressions in the left hand side of (4.35) and using (4.32) and (4.34), then from Cauchy-Schwartz inequality, one gets the following estimate

$$\begin{aligned} |\int_Q q(t,x)\, a^2(t,x)\, dx\, dt| &\leq \|(\mathscr{R}^1_{q_2} - \mathscr{R}^1_{q_1})(f_\lambda)\|_{L^2(\Sigma)} \|v_\lambda\|_{L^2(\Sigma)} + \frac{C}{\lambda}\|\varphi\|^2_{H^3(\mathbb{R}^n)} \\ &\quad + \|(\mathscr{R}^2_{q_2} - \mathscr{R}^2_{q_1})(f_\lambda)\|_{L^2(\Omega)} \|\partial_t v_\lambda(T,.)\|_{L^2(\Omega)} \\ &\quad + \|(\mathscr{R}^3_{q_2} - \mathscr{R}^3_{q_1})(f_\lambda)\|_{L^2(\Omega)} \|v_\lambda(T,.)\|_{L^2(\Omega)}. \end{aligned}$$

Then we obtain,

$$\begin{aligned} |\int_Q q(t,x)\, a^2(t,x)\, dx\, dt| \leq & \Big(\|(\mathscr{R}^1_{q_2} - \mathscr{R}^1_{q_1})(f_\lambda)\|^2_{L^2(\Sigma)} + \|(\mathscr{R}^2_{q_2} - \mathscr{R}^2_{q_1})(f_\lambda)\|^2_{H^1(\Omega)} + \|(\mathscr{R}^3_{q_2} - \mathscr{R}^3_{q_1})(f_\lambda)\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}} \\ & \Big(\|v_\lambda\|^2_{L^2(\Sigma)} + \|v_\lambda(T,.)\|^2_{L^2(\Omega)} + \|\partial_t v_\lambda(T,.)\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}} + \frac{C}{\lambda}\|\varphi\|^2_{H^3(\mathbb{R}^n)}. \end{aligned} \tag{4.36}$$

Setting

$$\phi_\lambda = (v_{\lambda|\Sigma},\, v_\lambda(T,.),\, \partial_t v_\lambda(T,.))$$

Then, from (4.36), we get

$$|\int_Q q(t,x) a^2(t,x)\, dx\, dt| \leq \|(\mathscr{R}_{q_2} - \mathscr{R}_{q_1})(f_\lambda)\|_{L^2(\Sigma)\times H^1(\Omega)\times L^2(\Omega)}\ \|\phi_\lambda\|_{L^2(\Sigma)\times L^2(\Omega)\times L^2(\Omega)} + \frac{C}{\lambda}\|\varphi\|^2_{H^3(\mathbb{R}^n)}$$

and using the fact that $f_\lambda(t,x) = u_{2,\lambda}(t,x)$ on Σ, we obtain

$$|\int_Q q(t,x) a^2(t,x)\, dx\, dt| \leq \|\mathscr{R}_{q_2} - \mathscr{R}_{q_1}\|\, \|u_{2,\lambda}\|_{H^1(\Sigma)}\, \|\phi_\lambda\|_{L^2(\Sigma)\times L^2(\Omega)\times L^2(\Omega)} + \frac{C}{\lambda}\, \|\varphi\|^2_{H^3(\mathbb{R}^n)},$$

Further, as $R_{i,\lambda|\Sigma} = 0$, for $i = 1, 2$, we deduce that

$$|\int_Q q(t,x)\, a^2(t,x)\, dx\, dt| \leq C \left(\|\mathscr{R}_{q_2} - \mathscr{R}_{q_1}\|\, \|u_{2,\lambda} - R_{2,\lambda}\|_{H^2(Q)}\, \|\phi_{1,\lambda}\|_{H^1(Q)\times L^2(\Omega)\times L^2(\Omega)} + \frac{1}{\lambda}\, \|\varphi\|^2_{H^3(\mathbb{R}^n)} \right),$$

where

$$\phi_{1,\lambda} = (v_\lambda - R_{1,\lambda},\, v_\lambda(T,.),\, \partial_t v_\lambda(T,.)).$$

Using the fact that $R_{1,\lambda}(T,.) = \partial_t R_{1,\lambda}(T,.) = 0$ on Ω, we have

$$\|u_2 - R_2\|_{H^2(Q)} \leq C\lambda^2\, \|\varphi\|_{H^3(\mathbb{R}^n)},$$

and

$$\begin{aligned} \|\phi_{1,\lambda}\|_{H^1(Q)\times L^2(\Omega)\times L^2(\Omega)} &\leq \|v_\lambda - R_{1,\lambda}\|_{H^1(Q)} + \|v_{\lambda|t=T}\|_{L^2(\Omega)} + \|\partial_t v_{\lambda|t=T}\|_{L^2(\Omega)} \\ &\leq C\lambda\, \|\varphi\|_{H^3(\mathbb{R}^n)}, \end{aligned}$$

Therefore by extending $q(t,x)$ by zero outside Q_r and recalling (2.4), we find out that

$$|\int_0^T \int_{\mathbb{R}^n} q(t, x - t\omega)\, \varphi^2(x)\, dx\, dt| \leq C \left(\lambda^3 \|\mathscr{R}_{q_2} - \mathscr{R}_{q_1}\| + \frac{1}{\lambda} \right) \|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

This completes the proof of the Lemma. □

Let's move now to prove the following Lemma

Lemma 4.2 *There exists a constant $C > 0$, $\beta > 0$, $\delta > 0$, and $\lambda_0 > 0$ such that for all $\omega \in \mathbb{S}^{n-1}$, we have*

$$|R(q)(\omega, y)| \le C \left(\lambda^{\beta} \|\mathscr{R}_{q_2} - \mathscr{R}_{q_1}\| + \frac{1}{\lambda^{\delta}} \right), \quad \text{a.e. } y \in \mathbb{R}^n.$$

for any $\lambda \ge \lambda_0$.

Proof . We consider $(\varphi_\varepsilon)_\varepsilon$ defined in the proof of Lemma 3.2. We only assume that $y \notin \Omega$, then for sufficiently small $\varepsilon > 0$, we can verify that supp $\varphi_\varepsilon \cap \Omega = \emptyset$. Taking in acount this last remark, using Lemma 4.1 and repeating the arguments used in Lemma 3.2, we obtain this estimate

$$|\int_{\mathbb{R}} q(t, y - t\omega)\, dt| \le C \left(\lambda^{\beta} \|\mathscr{R}_{q_2} - \mathscr{R}_{q_1}\| + \frac{1}{\lambda^{\delta}} \right), \quad \text{a.e. } y \notin B(0, \frac{r}{2}). \tag{4.37}$$

On the other hand, if $y \in B(0, \frac{r}{2})$, then we have

$$q(t, y - t\omega) = 0, \ \ \forall\, t \in \mathbb{R}. \tag{4.38}$$

Indeed, we have

$$|y - t\omega| \ge |t| - |y| \ge t - \frac{r}{2}. \tag{4.39}$$

So that, from (4.39), we deduce that for all $t > \frac{r}{2}$ we have $(t, y - t\omega) \notin \mathscr{C}_r^+$. And if $t \le \frac{r}{2}$, we have also that $(t, y - t\omega) \notin \mathscr{C}_r^+$. We recall that $Q_\sharp = Q \cap \mathscr{C}_r^+$. Consequently, we have

$$(t, y - t\omega) \notin Q_\sharp, \ \text{ for all } t \in \mathbb{R}.$$

Then, using the fact that $q = q_2 - q_1 = 0$ outside $Q_\sharp$, we obtain (4.38). Therefore

$$\int_{\mathbb{R}} q(t, y - t\omega)\, dt = 0, \quad \text{a.e. } y \in B(0, \frac{r}{2}). \tag{4.40}$$

In light of (4.37) and (4.40), the proof of Lemma 4.2 is completed. □

Using the above result and in the same way as in Section 3, we complete the proof of Theorem 2.

5 Proof of Theorem 3

In this section we deal with the same problem treated in Section 3 and 4, except our data will be the response of the medium for all possible initial data. As usual, we will prove Theorem 3 using geometric optics solutions constructed in Section 2 and X-ray transform. Let's first recall the definition of the operator $\mathcal{I}_q$:

$$\begin{array}{rcl} \mathcal{I}_q : H^1(\Sigma) \times H^1(\Omega) \times L^2(\Omega) & \longrightarrow & L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega). \\ \psi = (f, u_0, u_1) & \longmapsto & (\partial_\nu u,\, u(T, .),\, \partial_t u(T, .)). \end{array}$$

We denote by

$$\mathcal{I}^1_{q_j}(\psi) = \partial_\nu u_j, \ \ \mathcal{I}^2_{q_j}(\psi) = u_j(T, .), \ \ \mathcal{I}^3_{q_j}(\psi) = \partial_t u_j(T, .), \ \ \text{for } j = 1, 2.$$

Lemma 5.1 *Let $q_1,\, q_2 \in \mathcal{C}^1(\overline{Q})$, and put $q=(q_2-q_1)$. There exists $C>0$, $\beta>0$, $\delta>0$ and $\lambda_0>0$ such that for any $\omega\in\mathbb{S}^{n-1}$ we have the following estimate*

$$|R(q)(\omega,y)| \leq C\left(\lambda^{\beta}\|\mathcal{I}_{q_2}-\mathcal{I}_{q_1}\|+\frac{1}{\lambda^{\delta}}\right),\quad a.e.\, y\in\mathbb{R}^n.$$

for any $\lambda\geq\lambda_0$.

Proof . Let $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$. For λ sufficiently large, Lemma 2.2 guarantees the existence of the geometrical optics solution $u_{2,\lambda}$ to

$$(\partial_t^2-\Delta+q_2(t,x))u_{2,\lambda}(t,x)=0,\qquad \text{in } Q,$$

of the form

$$u_{2,\lambda}(t,x)=a(t,x)e^{i\lambda(x.\omega+t)}+R_{2,\lambda}(t,x) \tag{5.41}$$

where $R_{2,\lambda}$ satisfies

$$\partial_t R_{2,\lambda|t=0}=R_{2,\lambda|t=0}=0,\quad R_{2,\lambda|\Sigma}=0,$$

and

$$\|R_{2,\lambda}\|_{L^2(Q)}\leq\frac{C}{\lambda}\|\varphi\|_{H^3(\mathbb{R}^n)}. \tag{5.42}$$

We denote $u_{1,\lambda}$ the solution of

$$\left\{\begin{array}{ll}\big(\partial_t^2-\Delta+q_1(t,x)\big)\,u_{1,\lambda}(t,x)=0 & \text{in } Q,\\ u_{1,\lambda}(0,x)=u_{2,\lambda}(0,x),\;\;\partial_t u_{1,\lambda}(0,x)=\partial_t u_{2,\lambda}(0,x) & \text{in } \Omega,\\ u_{1,\lambda}(t,x)=u_{2,\lambda}(t,x):=f_\lambda(t,x), & \text{on } \Sigma.\end{array}\right.$$

Putting $u_\lambda(t,x)=u_{1,\lambda}(t,x)-u_{2,\lambda}(t,x)$, we get that

$$\left\{\begin{array}{ll}\big(\partial_t^2-\Delta+q_1(t,x)\big)\,u_\lambda(t,x)=q(t,x)u_{2,\lambda}(t,x) & \text{in } Q\\ u_\lambda(0,x)=\partial_t u_\lambda(0,x)=0 & \text{in } \Omega\\ u_\lambda(t,x)=0 & \text{on } \Sigma.\end{array}\right.$$

Applying Lemma 2.2, once more for λ large enough, we may find a geometrical optic solution v_λ to the backward wave equation

$$\big(\partial_t^2-\Delta+q_1(t,x)\big)\,v_\lambda(t,x)=0,\;\; \text{in } Q,$$

of the form

$$v_\lambda(t,x)=a(t,x)e^{-i\lambda(x.\omega+t)}+R_{1,\lambda}(t,x), \tag{5.43}$$

where $R_{1,\lambda}$ satisfies

$$\partial_t R_{1,\lambda|t=T}=R_{1,\lambda|t=T}=0,\quad R_{1,\lambda|\Sigma}=0,$$

and

$$\|R_{1,\lambda}\|_{L^2(Q)}\leq\frac{C}{\lambda}\,\|\varphi\|_{H^3(\mathbb{R}^n)}. \tag{5.44}$$

By integrating by parts and using the Green's formula, one gets

$$\int_Q q(t,x)\,u_{2,\lambda}(t,x)v_\lambda(t,x)\,dx\,dt=\int_\Sigma(\mathcal{I}_{q_2}^1-\mathcal{I}_{q_1}^1)(\psi_\lambda)v_\lambda(t,x)\,d\sigma\,dt$$

$$+\int_\Omega \left(\mathcal{I}_{q_2}^2-\mathcal{I}_{q_1}^2\right)(\psi_\lambda)\,\partial_t v_\lambda(T,.)\,dx-\int_\Omega \left(\mathcal{I}_{q_2}^3-\mathcal{I}_{q_1}^3\right)(\psi_\lambda)\,v_\lambda(T,.)\,dx, \qquad (5.45)$$

where

$$\psi_\lambda=(u_{2,\lambda|\Sigma},u_{2,\lambda|t=0},\partial_t u_{2,\lambda|t=0}).$$

Next, we proceed by a similar way as in the proof of Lemma 4.1, we get

$$\begin{aligned}|\int_Q q(t,x)a^2(t,x)\,dx\,dt| &\leq \|\mathcal{I}_{q_2}-\mathcal{I}_{q_1}\|\,\|\psi_\lambda\|_{H^1(\Sigma)\times H^1(\Omega)\times L^2(\Omega)}\|\phi_\lambda\|_{L^2(\Sigma)\times L^2(\Omega)\times L^2(\Omega)}\\ &\quad +\frac{C}{\lambda}\,\|\varphi\|^2_{H^3(\mathbb{R}^n)}\end{aligned}$$

where

$$\phi_\lambda=(v_{\lambda|_\Sigma},\,v_{\lambda|t=T},\,\partial_t v_{\lambda|t=T}).$$

Further, as $R_{i,\lambda|\Sigma}=0$, for $i=1,2,$, we deduce that

$$\begin{aligned}|\int_Q q(t,x)\,a^2(t,x)\,dx\,dt| &\leq \|\mathcal{I}_{q_2}-\mathcal{I}_{q_1}\|\,\|\psi_{1,\lambda}\|_{H^2(Q)\times H^1(\Omega)\times L^2(\Omega)}\|\phi_{1,\lambda}\|_{H^1(Q)\times L^2(\Omega)\times L^2(\Omega)}\\ &\quad +\frac{C}{\lambda}\,\|\varphi\|^2_{H^3(\mathbb{R}^n)},\end{aligned}$$

where

$$\phi_{1,\lambda}=\left(v_\lambda-R_{1,\lambda},\,v_{\lambda|t=T},\,\partial_t v_{\lambda|t=T}\right),\quad \psi_{1,\lambda}=\left(u_{2,\lambda}-R_{2,\lambda},\,u_{2,\lambda|t=0},\,\partial_t u_{2,\lambda|t=0}\right).$$

Using the fact that $R_{1,\lambda}(T,.)=\partial_t R_{1,\lambda}(T,.)=0$ on Ω, we have

$$\|\phi_{1,\lambda}\|_{H^1(Q)\times L^2(\Omega)\times L^2(\Omega)}\leq C\lambda\,\|\varphi\|_{H^3(\mathbb{R}^n)},$$

On the other hand, since $R_{2,\lambda}(0,.)=\partial_t R_{2,\lambda}(0,.)=0$ on Ω, we have

$$\begin{aligned}\|\psi_{1,\lambda}\|_{H^2(Q)\times H^1(\Omega)\times L^2(\Omega)} &\leq \|u_{2,\lambda}-R_{2,\lambda}\|_{H^2(Q)}+\|u_{2,\lambda|t=0}\|_{H^1(\Omega)}+\|\partial_t u_{2,\lambda|t=0}\|_{L^2(\Omega)}\\ &\leq C\lambda^2\,\|\varphi\|_{H^3(\mathbb{R}^n)},\end{aligned}$$

Therefore by extending $q(t,x)$ by zero outside Q and recalling (2.4), we find out that

$$|\int_0^T\int_{\mathbb{R}^n} q(t,x-t\omega)\,\varphi^2(x)\,dx\,dt|\leq C\left(\lambda^3\|\mathcal{I}_{q_2}-\mathcal{I}_{q_1}\|+\frac{1}{\lambda}\right)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

Now, in order to complete the proof of Lemma 5.1, it will be enough to fix $y\in\mathbb{R}^n$, consider $(\varphi_\varepsilon)_\varepsilon$ defined as before, and proceed as in the proof of Lemma 3.2. By repeating the arguments used in the previous sections, we complete the proof of Theorem 3. $\square$

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